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Classification of First-Order Lagrangians on the Line

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We present a complete classification for first-order Lagrangians defined on the line according to the Noether point symmetry algebra they admit. All possible canonical forms of Lagrangians that admit Noether algebras are given.

1. INTRODUCTION

The simplest equivalence problem in the calculus of variations or in Lagrangian mechanics is to determine under what conditions two scalar variational problems of first order are transformable into each other by means of a suitable point transformation.

This problem has been investigated by many researchers using the Cartan method (see, e.g., Kamran and Olver, 1989). Here we use the symmetry method which involves investigating the (Noether) point symmetry algebras associated with Lagrangians. The objective here is to determine all canonical forms for particle Lagrangians defined on the line. After doing so, we make a synthesis of all first-order Lagrangians that admit a symmetry algebra. While some work has been done in this regard (Kara *et al.*, 1994; Adam *et al.*, 1994), there has been no mention of a complete classification of such Lagrangians. The results here have applications in mechanics.

In Section 2, we discuss canonical forms for first-order Lagrangians corresponding to scalar linearizable ordinary Euler–Lagrange equations. In

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Section 3, we present the general Noether equivalence problem. Finally, in Table II of Section 4, we provide a complete classification of first-order Lagrangians on the line.

Below we outline a few definitions and results that will be useful for what follows. Although what follows applies for nth-order Lagrangians, we specialize to first-order Lagrangians in the sequel.

1.1. Euler–Lagrange Equations and the Inverse Problem

Consider a *variational problem* where the extremal of the functional

$$\int_\Omega L(t, q^{[n]}(t))\, dt \tag{1.1}$$

is desired, in which Ω is some interval containing t and $q^{[n]}(t)$ is the tuple which contains all the derivatives of scalar q with respect to t up to order n. The solution of the above problem is related to the solution of a $2n$th-order differential equation by way of the Euler–Lagrange operator $\mathscr{E}$ defined below.

Definition 1.1. The *Euler–Lagrange operator* is defined by

$$\mathscr{E} = \sum_{j=0}^{n} \left(-\frac{d}{dt}\right)^j \frac{\partial}{\partial q^{(j)}} \tag{1.2}$$

where $q^{(j)}$ is the jth total derivative of q with respect to t.

The integrand of the functional L in (1.1) is referred to as the *Lagrangian*. The Euler–Lagrange equation corresponding to $L(t, q^{[n]}(t))$ is an ordinary differential equation of order $2n$ and is given by

$$\mathscr{E}(L) = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial q^{(1)}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial q^{(2)}} - \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial L}{\partial q^{(n)}} = 0$$

The *inverse problem* in the calculus of variations involves determining a Lagrangian L corresponding to a given differential equation (if such a Lagrangian exists). That is, given a differential equation $E(t,q^{[n]}(t)) = 0$, find a Lagrangian L such that the Euler–Lagrange equation $\mathscr{E}(L) = 0$ is equivalent to the given differential equation.

1.2. Invariance of the Lagrangian

In this section, we summarize the relevant aspects of the invariance of the functional (1.1) or, equivalently, the symmetries associated with the respective Lagrangian (in short, we shall say symmetries of the Lagrangian or simply Noether symmetries).

For real-valued $q(t)$, we may induce a change of variables

$$T = H(t, q), \qquad Q = F(t, q)$$

A change of variables in the derivative obtained by prolongation is then also induced so that the functional (1.1) is then transformed into a corresponding functional $\int_{\bar{\Omega}} \bar{L}(T, Q^{[n]}(T))\, dT$, where $\bar{\Omega} = \{T, t \in \Omega\}$. This will be the case if

$$L(t, q^{[n]}(t)) = \bar{L}(T, Q^{[n]}(T)) \frac{dT}{dt} + \frac{df}{dt} \tag{1.3}$$

by means of the prescribed change of variables, where f, the gauge term, is a function of t and q. The two Lagrangians L and $\bar{L}$ are said to be *equivalent up to gauge*. This means that the Euler–Lagrange equations associated with L and $\bar{L}$ are equivalent to each other by the said change of variables. For strict invariance we require $f = 0$.

The *point* symmetries associated with a Lagrangian $L(t, q^{[n]}(t))$ are of the form

$$G = \xi(t, q) \frac{\partial}{\partial t} + \eta(t, q) \frac{\partial}{\partial q} \tag{1.4}$$

defined on $\mathbb{R}^2$. On the prolonged space, viz., the $(t, q^{[n]})$ space, the nth prolongation or extension of G is

$$G^{[n]} = G + \sum_{i=1}^{n} \zeta_i \frac{\partial}{\partial q^{(i)}}$$

where

$$\zeta_1 = \frac{d\eta}{dt} - q^{(1)} \frac{d\xi}{dt}, \qquad \zeta_{j+1} = \frac{d\zeta_j}{dt} - q^{(j+1)} \frac{d\xi}{dt} \qquad \text{for} \quad j = 1, \ldots, n-1$$

The following theorem provides a mechanism for determining the Noether symmetries G associated with a Lagrangian L.

Theorem 1.1. G as in (1.4) is a point symmetry of a Lagrangian L if and only if

$$G^{[n]}L + L \frac{d\xi}{dt} = \frac{df}{dt} \tag{1.5}$$

Equation (1.5) is often referred as a *Killing-type* equation (Sarlet and Cantrijn, 1981).

1.3. Alternative Lagrangians

In this section, we introduce notions which takes into account the non-uniqueness of the Lagrangian description of a given differential equation.

(Although the definition of alternative Lagrangians given below is applicable, little has been done in the characterization of these Lagrangians for partial differential equations.)

Definition 1.2. Two Lagrangians L and $\tilde{L}$ are said to be *alternative* Lagrangians for a given differential equation if their respective Euler–Lagrange equations imply each other, i.e., $\mathscr{E}(L) = 0$ if and only if $\mathscr{E}(\tilde{L}) = 0$.

In the case of first-order Lagrangians, it can be shown that a given second-order dynamical equation can always be cast as an Euler–Lagrange equation for an infinitude of Lagrangians all of which need not have the same number of Noether point symmetries. Hence, a variety of nontrivial definitions concerning equivalent Lagrangians associated with a given equation can be found in the literature.

2. CANONICAL FORMS FOR LINEARIZABLE EQUATIONS

The most general scalar ordinary second-order equation which is linearizable by means of a point transformation is given by

$$\ddot{q} = A(t, q)\dot{q}^3 + B(t, q)\dot{q}^2 + C(t, q)\dot{q} + D(t, q) \tag{2.1}$$

where A, B, C, and D satisfy the conditions

$$3A_{tt} + 3A_tC - 3A_qD + 3AC_t + C_{qq} - 6AD_q + BC_q - 2BB_t - 2B_{tq} = 0$$

$$6A_tD - 3B_qD + 3AD_t + B_{tt} - 2C_{tq} - 3BD_q + 3D_{qq} + 2CC_q - CB_t = 0$$

(see Mahomed and Leach, 1989a). Thus, a Lagrangian for (2.1) is obtained by using (1.3) with $\bar{L}$ being the Lagrangian of the simplest linear equation $Q'' = 0$. For example, we can choose the simplest Lagrangian $\bar{L} = \frac{1}{2}Q'^2$. Hence, instead of constructing Lagrangians for (2.1), we may, without loss of generality, construct Lagrangians for

$$\ddot{q} = 0 \tag{2.2}$$

We now consider the most general Lagrangian of (2.2), viz.,

$$L' = \int^{\dot{q}}\int^{r} g(s, q - ts)\, ds\, dr + H_1(t, q)\dot{q} + J(t, \dot{q}) \tag{2.3}$$

where g is an arbitrary function of the first integrals of $\ddot{q} = 0$ and $H_{1t} = J_q$ (Mahomed *et al.*, 1993). The classification according to the Noether point symmetry algebra of symmetries for (2.3) has been investigated in Mahomed *et al.* (1993). We recall that if G' is a symmetry of (2.3), then it is of the form

$$G' = (a_1 + a_2t + a_3q + a_4tq + a_5t^2)\frac{\partial}{\partial t} + (a_4q^2 + a_5tq + a_6 + a_7q + a_8t)\frac{\partial}{\partial q} \tag{2.4}$$

where the a_i are arbitrary constants. The function g satisfies

$$(a_6 - a_1\dot{q} - a_3\dot{q}v + a_4v + a_7v)\frac{\partial g}{\partial v} + (2a_7 - a_2 - 3a_3\dot{q} + 3a_4v)g + (a_8 + a_7\dot{q} - a_2\dot{q} + a_5v - a_3\dot{q}^2 + a_4\dot{q}v)\frac{\partial g}{\partial \dot{q}} = 0 \tag{2.5}$$

where $v = q - t\dot{q}$. The possible forms of g for which (2.3) admits a Noether algebra are given in Adam *et al.* (1994). The corresponding Lagrangian for this case is listed in Table II. As a consequence of this classification, the following result is proved in the above reference.

Theorem 2.1. A Lagrangian of the form (2.3) admits a maximal real one-, two-, three-, or five-dimensional Noether point symmetry algebra.

It is well known that a linearizable second-order equation (2.1) admits the maximal eight-dimensional Lie algebra $sl(3, \mathbb{R})$. In contrast, a Lagrangian of the *linearizable* second-order equation can admit any one of one-, two-, three-, or five-dimensional Noether point symmetry algebra. Hence, a variational formulation is richer in terms of an algebraic analysis. This is due to the alternative Lagrangian (2.3) of (2.2).

3. THE GENERAL NOETHER EQUIVALENCE PROBLEM

We now consider the general case, i.e., when a second-order equation is not linearizable by means of a point transformation. The discussion here depends on the following theorem proved in Mahomed and Leach (1989b).

Theorem 3.1. A scalar second-order ordinary differential equation does not admit a maximal real four-, five-, six-, or seven-dimensional point symmetry algebra.

The classification of Lagrangians according to Noether point symmetries is made by considering the Lie equivalence problem. The complete real classification of second-order ordinary differential equations according to the Lie point symmetry algebra they admit is given in Mahomed and Leach (1989b) and Mahomed *et al.* (1993). We summarize the nonlinearizable equations in Table I together with the point symmetries they admit. Lagrangian representations of these equations are given in Table II.

We refer to the Lie algebras in Table I as $L_{i;j}$ if there is more than one algebra L_i of dimension i. It has been shown in Kara *et al.* (1994) that if a scalar second-order ordinary differential equation admits a maximal one-, two-, or three-dimensional algebra $\mathcal{A}$, then there exists a Lagrangian which admits $\mathcal{A}$ as its Noether algebra provided $\mathcal{A}$ is the real algebra L_1, $L_{2;1}$, $L_{2;2}$, $L_{3;2}$ ($a = -1$), $L_{3;3}$ ($b = 0$), $L_{3;4}$, $L_{3;5}$, or $L_{3;6}$. However, if an equation admits $L_{3;1}$, $L_{3;2}$ (a arbitrary), or $L_{3;3}$ (b arbitrary) as its maximal algebra, then the Noether algebra is at most two dimensional.

4. DISCUSSION

This paper completely classifies first-order Lagrangians associated with scalar second-order ordinary differential equations admitting non-similar Noether point symmetry algebras. We remind the reader that two Noether algebras are similar if they are isomorphic to each other and they can be transformed into each other by a point transformation using (1.3). We summarize the results in Table II. For the case in which the Euler–Lagrange equation admits the $sl(3, \mathbb{R})$ algebra, we list the functions g from which the Lagrangian is obtained by (2.3) subject to the condition $H_{1t} = J_q$ (note that $v = q - t\dot{q}$). For the other cases we list the Lagrangian L.

We note that in Table II, A is a nonzero constant and α an arbitrary nonzero function of its argument in each case. Also, Table II presents a real classification of particle Lagrangians. In the complex classification there are seven nonisomorphic three-dimensional algebras and one no longer requires the 5th, 11th, and 13th cases of dimension three in Table II.

Table I. Nonlinearizable Equations and Point Symmetries[a]

Dimension	Symmetries	Equation
L_1	r	$\ddot{q} = E(t, \dot{q})$
L_2	p, r	$\ddot{q} = E(\dot{q})$
	$p,\ tp + qr$	$t\ddot{q} = E(\dot{q})$
L_3	$r,\ p,\ tp + (t + q)r$	$\ddot{q} = \exp(-\dot{q})$
	$p,\ r,\ tp + aqr$	$\ddot{q} = \dot{q}^{(a-2)/(a-1)}$ ($a \neq 0, 1/2, 1, 2$)
	$p,\ r(bt + q)p + (bq - t)r$	$\ddot{q} = (1 + \dot{q}^2)^{3/2} \exp(b \arctan \dot{q})$
	$r,\ tp + qr,\ 2tqp + q^2 r$	$t\ddot{q} = A\dot{q}^3 - \frac{1}{2}\dot{q}$
	$r,\ tp + qr,\ 2tqp + (q^2 \mp t^2)r$	$t\ddot{q} = \pm \dot{q}^3 + \dot{q} + A(1 \pm \dot{q}^2)^{3/2}$
	$(1 + t^2)p + tqr,\ tqp + (1 + q^2)r,\ qp - tr$	$\ddot{q} = A\left[\dfrac{1 + \dot{q}^2 + (q - t\dot{q})^2}{1 + t^2 + q^2}\right]^{3/2}$

[a] $p = \partial/\partial t$, $r = \partial/\partial q$.

Table II. Lagrangian Representations[a]

Dimension	Symmetry	Lagrangian/g
1	r	$\int^{\dot q} F(t, x)\, dx\ (\dot F = 0)$
	$tp + qr$	$g = \alpha(\dot q)/v$
	$tp + aqr,\ a \neq 0, 1$	$g = \dot q^{(1-2a)/(a-1)}\alpha(v\dot q^{a/(1-a)})$
	$p + tr$	$g = \alpha(v + \frac{1}{2}\dot q^2)$
	$p + qr$	$g = \alpha[\dot q \exp(v/\dot q)]/\dot q^2$
2	p, r	$\int^{\dot q} [\int^r ds/\mathcal{H}(s)]\, dr - t\dot q$
	$r, tp + qr$	$(1/t) \int^{\dot q} [\exp \int^r ds/\mathcal{H}(s)]\, dr$
	$r, qr + p$	$g = A\dot q^{-2}$
	p, r	$g = \alpha(\dot q)$
	tp, qr	$g = A/(v\dot q)$
3	$p, r, tp - qr$	$-4\dot q^{1/2} + q$
	$p, r, qp - tr$	$-(1 + \dot q^2)^{3/2} + t\dot q$
	$p, tp + qr, 2tqp + q^2r$	$\dot q/t + 1(2t\dot q)$ or $\dot q/t - 1/(2t\dot q)$
	$r, tp + qr, 2tqp + (q^2 \mp t^2)r$	$t^{-1}(1 \pm \dot q^2)^{1/2} + At^{-1}\dot q$
	$(1 + t^2)p + tqr, tqp + (1 + q^2)r,$ $qp - tr$	([b])
	$r, p, tp + (t + q)r$	$g = A \exp(-\dot q)$
	$p, r, tp - qr$	$g = A\dot q^{-3/2}$
	$p, r, tp + aqr\ [a = (1 + s)/(2 + s)]$	$g = A\dot q^s,\ s \neq 0, -2, -3$
	p, r, tp	$g = A\dot q^{-1}$
	$p, r, qp - tr$	$g = A(1 + \dot q^2)^{-3/2}$
	$p, r, (bt + q)p + (bq - t)r,\ b > 0$	$g = A(1 + \dot q^2)^{-3/2} \exp(b \arctan \dot q)$
	$r, tp + qr, 2tqp + (q^2 \mp t^2)r$	$g = A(q - t\dot q \pm \dot q^2/2)^{-3/2}$
	$(1 + t^2)p + tqr, tqp + (1 + q^2)r,$ $qp - tr$	$g = A[1 + (q - t\dot q)^2 + \dot q^2]^{-3/2}$
5	$p, r, tr, t^2p + tqr, 2tp + qr$	$\frac{1}{2}\dot q^2$

[a] $p = \partial/\partial t$, $r = \partial/\partial q$.

[b] Here the Lagrangian is of the form

$$L = \frac{1}{x^2}(1 + t^2)^{-3/2}[(y - \dot q)\sin \omega + \beta \sec \omega] + b(t, q)\dot q + c(t, \dot q)$$

where b and c satisfy $b_t = c_q + A(1 + t^2 + q^2)^{-3/2}$ and $x = (1 + t^2 + q^2)^{1/2}/(1 + t^2)$, $\beta = qx$, $\tan \omega = (\dot q - y)/x$, and $y = tq/(1 + t^2)$.

Some remarks are now in order. First, and this is an important point peculiar to Noether algebras, two Noether algebras can be similar but their respective Lagrangians need not be equivalent to each other. This can easily be observed from Table II. To illustrate this, we consider the algebra of symmetries p, r, $tp - qr$. This representation implies two Lagrangians (see Table II) which are not equivalent to each other by means of a point transformation. Second, we make the point that the dimensionality of the Noether algebra of a Lagrangian associated with a linear or linearizable equation need not be maximal (i.e., five dimensional). This was apparent in our discussion on alternative Lagrangians.

REFERENCES

Adam, A. A., Mahomed, F. M., and Kara, A. H. (1994). *Quaestiones Mathematicae,* **17**(4) 469.

Kamran, N., and Olver, P. J. (1989). *J. Diff. Eq.*, **80**, 32.

Kara, A. H., Mahomed, F. M., and Leach, P. G. L. (1994). Noether equivalence problem for particle Lagrangians, *Journal of Mathematical Analysis and Applications,* **188**, 867.

Mahomed, F. M., and Leach, P. G. L. (1989a). *Journal of Mathematical Physics,* **30**, 2770.

Mahomed, F. M., and Leach, P. G. L. (1989b). *Quaestiones Mathematicae,* **12**, 121.

Mahomed, F. M., Kara, A. H., and Leach, P. G. L. (1993). *Journal of Mathematical Analysis and Applications,* **178**, 116.

Sarlet, W., and Cantrijn, F. (1981). *SIAM Rev.*, **23**, 467.